(Too) Strong Convergence in New Growth and Diffusion Models

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Five years ago, a new type of endogenous growth/dispersion model was born [Alvarez et al., 2008, Lucas, 2008]. Building on important older work [Kortum, 1997, Eaton and Kortum, 2004], theorists built a model in which human capital was not quite external, and not quite internal. These models are beautiful. People meet, exchange ideas, and the economy pulls itself up by its own bootstraps. The long-run growth path of productivity distributions in an economy can be described by only two or three parameters. Balanced growth paths feature a productivity distribution which smoothly wanders up the real line.

I became interested in this class of models because they imply that migration is important. Immigrants and return migrants can be the vessels which take ideas from one place to another. In a couple of notes on my website I write down such models formally [Jinkins, 2012a,b]. As I delved deeper into this class of model, however, I found that it was difficult to find a balanced growth path where countries both interact with each other and also grow at different rates.[1] As Pritchett [1997] argues, there are places in the world that are not only very poor relative to other places, but also that remain stagnant even as their distance from the technology frontier grows. This is a feature which we would like to come naturally from a growth model.

In this note, I will focus on the model presented in Alvarez et al. [2011](hereafter ABL). I will show that even a very brief and severely limited exposure to foreign ideas will drastically change a poor country's long term growth prospects, so that its ultimate growth rate will match that of the rich country. Growth in this class of model is ultimately determined by the tail of the productivity distribution, and even a tiny exposure to foreign ideas is enough to steal another country's tail.

Model

First I will focus on a single country. Below I basically restate the model, including notation, from ABL. There are a continuum of goods indexed by the unit interval. Utility is CES, with elasticity η:

$$U(C) = \left(\int_0^1 c(i)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}} di$$

Production technology is linear, with unit labor requirements for each good i given by $z(i)$. The equilibrium concept is competitive equilibrium, so that the price of goods is just unit cost: $wz(i)$. Reordering goods by labor requirements, let F(z,t) be the right CDF of goods, i.e. the percentage of goods with labor requirement higher than z. F is defined on the non-negative real line. GDP is wage divided by the ideal price index:

[1]In the example I discuss below, if different countries have different meeting technologies, long-run growth rates will be different. However, this is because I assume that there is a short communication between countries, and then countries are isolated again. The next step in this paper is to show that if communication is permanent, then even with different meeting technologies, countries will have the same growth rate. This is because one country becomes so much better than the other, than even a small number of draws from the good countries distribution has a large effect on the bad country's productivity. Meeting technology will have an effect on long run GDP *levels*, but not on growth *rates*.

$$y(t) = \left(\int_0^\infty z^{\frac{\eta-1}{\eta}} f(z,t) dz \right)^{\frac{-\eta}{\eta-1}} \quad (1)$$

The first important growth parameter is α, the number of independent draws which each good gets per unit of time[2]. Following ABL, let draws be taken from the right CDF G(z,t). If good 1 "meets" a good with a lower unit labor cost, then good 1 can now be produced with the lower unit labor cost. Consider a time period Δ:

$$F(z, t + \Delta) = F(z, t) \left(G(z, t) \right)^{\alpha \Delta}$$

Taking logs and letting Δ approach zero, we get:

$$\frac{\partial \ln F(z,t)}{\partial t} = \alpha \ln G(z,t) \quad (2)$$

Integrating both sides, and plugging the initial condition:

$$\ln F(z,t) = \ln F(z,0) + \alpha \int_0^t \ln G(z,s) ds$$

Now, suppose that $G(z,t) = F(z,t)$, so that draws are taken from an economy's own distribution. Then we can solve (2):

$$\ln F(z,t) = \ln F(z,0) e^{\alpha t} \quad (3)$$

We will be interested in balanced growth paths, defined in this class of models as a parameter ν and a distribution Φ such that:

$$F(z,t) = \Phi(e^{\nu t} z) \quad (4)$$

A glance at (1) shows that the growth rate of the economy on a balanced growth path will be ν. Plugging (4) into (3) we get:

$$\ln \Phi(e^{\nu t} z) = \ln \Phi(z) e^{\alpha t} \quad (5)$$

ABL prove that the family of right CDF's which satisfy (5) is Weibull, described by two parameters–a scale parameter λ and a shape parameter θ:

$$F(z,0) = e^{-\lambda z^{\frac{1}{\theta}}}$$

Furthermore, the GDP growth rate is $\nu = \alpha\theta$. We can think of θ as summarizing how fat the tail of productivity is, so that the growth rate depends on both the fatness of the tail and the number of draws per period.

The last result from ABL I will mention is on stability. A labor requirement distribution F is said to converge to a BGP with Weibull parameters θ and λ if:

[2] In the first generation of these growth papers, α was described as the number of meetings each *person* gets per period. In ALM they do it as the number of meetings each *good* gets per period. The reason is that ALM want firms to be able to scale. A single person can only provide a single unit of labor, and in the original models cannot share his productivity. In the new models, firms can hire labor to use their technology, which is per good, and is available to everyone in the economy. The "hard to transfer human capital" interpretation I gave these models earlier is based on the meetings per person interpretation. I find it more understandable to think about α this way.

$$\lim_{t\to\infty} \ln\left[F\left(e^{-\alpha\theta t}z,t\right)\right] = -\lambda z^{\frac{1}{\theta}} \tag{6}$$

If the initial distribution satisfies the following property for parameters $n > 0$ and $K > 0$, then (6) holds for $\theta = n$ and $\lambda = K$:

$$\lim_{z\to 0} nf(z^n,0)z^{n-1} = K \tag{7}$$

Tail-Stealing

In this section, I demonstrate the sensitivity of growth rates in ABL with an example. Let there be two countries, called India and England. Let India have an initial labor requirement distribution F(z,0) which satisfies (7) with parameters θ and λ, and let England have a distribution G(z,0) which satisfies (7) with parameters γ and μ. Let both countries have the same meeting rate α. In autarchy, India and England will have respective BGP GDP growth rates of $\alpha\theta$ and $\alpha\gamma$.

Initially for a period Δ, we will let Indian goods draw from the English distribution with probability β, and their own distribution with probability α. Using (2), we get:

$$\ln F(z,\Delta) = e^{\alpha\Delta}\ln F(z,0) + \beta\int_0^\Delta \ln G(z,s)ds$$

Exponentiating, taking derivatives, and setting $A = e^{\alpha\Delta}$:

$$f(z,\Delta) = AF(z,0)^{A-1}f(z,0) + e^{\beta\int\ln Gds}\beta\int_0^\Delta \frac{g(z,s)}{G(z,s)}ds$$

Now we take $f(z,\Delta)$ to be the new "initial" Indian distribution, and find parameters for which (7) is satisfied. First note that both $F(0,0)$ and $G(0,0)$ are equal to one, and we will assume that they approach the limit smoothly. Guess first that $n = \gamma = m\theta$. Then:

$$\begin{aligned}\lim_{z\to 0} A\gamma f(z^\gamma,0)z^{\gamma-1} &= Am\lim_{z\to 0}\theta f((z^m)^\theta,0)(z^m)^{\theta-1}(z^m)^{1-\frac{1}{m}}\\ &= Am\lambda\lim_{z\to 0}(z^m)^{1-\frac{1}{m}}\\ &= 0\end{aligned} \tag{8}$$

I may have to rule out some crazy distributions in order to switch the limit and integral below:

$$\begin{aligned}\lim_{z\to 0}\beta\gamma z^{\gamma-1}\int_0^\Delta g(z^\gamma,s)ds &= \beta\lim_{z\to 0}\int_0^\Delta \gamma z^{\gamma-1}g(z^\gamma,s)ds\\ &= \beta\int_0^\Delta \lim_{z\to 0}\gamma z^{\gamma-1}g(z^\gamma,s)ds\\ &= \beta\int_0^\Delta e^{\alpha s}\mu ds\\ &= \beta\left(e^{\alpha\Delta}-1\right)\mu\\ &= B\mu\end{aligned} \tag{9}$$

Combining (8) and (9), we see that $f(z,\Delta)$ now satisfies (7) with parameters γ and $B\mu$. Therefore, the Indian distribution will converge to a Weibull labor requirement distribution with the same BGP growth rate as England. In effect, India has stolen England's tail. Moreover, India's tail parameter γ, and thus long-run growth rate, does not depend on the length of exposure time Δ or the rate of drawing from the foreign distribution β, so long as both parameters are positive.

Bounding Convergence Time

Consider the special case in which India and England are already on their balanced growth paths. India's initial right cost CDF is then:

$$F(z,0) = e^{-\lambda z^{\frac{1}{\theta}}}.$$

England's initial right CDF is:

$$G(z,0) = e^{-\mu z^{\frac{1}{\gamma}}}.$$

As before, let $\gamma > \theta$. Let $C = \lambda e^{\alpha\Delta}$, and $D = \beta\mu\left(e^{\alpha\Delta} - 1\right)$. Performing the same intervention as above, we can rewrite the GDP equation (1) as:[3]

$$\begin{aligned} y(t) &= \left(\int_0^\infty z^{\frac{\eta-1}{\eta}}\left(Ce^{\alpha t}\frac{1}{\theta}z^{\frac{1-\theta}{\theta}} + De^{\alpha t}\frac{1}{\gamma}z^{\frac{1-\gamma}{\gamma}}\right)e^{-\left(Ce^{\alpha t}z^{\frac{1}{\theta}} + De^{\alpha t}z^{\frac{1}{\gamma}}\right)}dz\right)^{\frac{-\eta}{\eta-1}} \\ &= e^{\alpha\gamma t}\left(\int_0^\infty x^{\frac{\eta-1}{\eta}}\left(Ce^{\alpha(1-\frac{\gamma}{\theta})t}\frac{1}{\theta}x^{\frac{1-\theta}{\theta}} + D\frac{1}{\gamma}x^{\frac{1-\gamma}{\gamma}}\right)e^{-\left(Ce^{\alpha(1-\frac{\gamma}{\theta})t}x^{\frac{1}{\theta}} + Dx^{\frac{1}{\gamma}}\right)}dx\right)^{\frac{-\eta}{\eta-1}} \end{aligned} \quad (10)$$

As is clear from (10) and consistent with the last section, the Indian economy is converging to GDP:

$$\begin{aligned} y_\infty(t) &= e^{\alpha\gamma t}\left(\int_0^\infty x^{\frac{\eta-1}{\eta}}D\frac{1}{\gamma}x^{\frac{1-\gamma}{\gamma}}e^{-Dx^{\frac{1}{\gamma}}}dx\right)^{\frac{-\eta}{\eta-1}} \\ &= e^{\alpha\gamma t}\left(\frac{1}{\gamma}D^{-\gamma\frac{\eta-1}{\eta}}\Gamma\left[1+\gamma\frac{\eta-1}{\eta}\right]\right)^{\frac{-\eta}{\eta-1}} \end{aligned} \quad (11)$$

One might ask how long it will take the economy to approach within ε% of its long run GDP. Given any $\varepsilon > 0$, my goal is to find a time T such that for all $t > T$:

$$\frac{y_\infty(t) - y(t)}{y_\infty(t)} < \varepsilon. \quad (12)$$

We can separate (10) into two terms:

$$\begin{aligned} y(t) = e^{\alpha\gamma t}\Bigg(&\int_0^\infty x^{\frac{\eta-1}{\eta}}Ce^{\alpha(1-\frac{\gamma}{\theta})t}\frac{1}{\theta}x^{\frac{1-\theta}{\theta}}e^{-\left(Ce^{\alpha(1-\frac{\gamma}{\theta})t}x^{\frac{1}{\theta}} + Dx^{\frac{1}{\gamma}}\right)} \\ &+ \int_0^\infty x^{\frac{\eta-1}{\eta}}D\frac{1}{\gamma}x^{\frac{1-\gamma}{\gamma}}e^{-\left(Ce^{\alpha(1-\frac{\gamma}{\theta})t}x^{\frac{1}{\theta}} + Dx^{\frac{1}{\gamma}}\right)}\Bigg)^{\frac{-\eta}{\eta-1}} \end{aligned} \quad (13)$$

[3] I drop the Δ from the intervention to make notation simpler, but all t's refer to time after the intervention is finished.

Two terms are added inside the parenthesis of (13). As $t \to \infty$, the first term goes to zero from above, and the second term goes to the interior of (11) from below. Since $y(t)$ approaches $y_\infty(t)$ from below,[4] I will ignore the second term. Define:

$$\psi \equiv \frac{1}{1-\varepsilon}^{\frac{\eta}{\eta-1}} - 1, \tag{14}$$

It can be shown that (12) is satisfied if the first term of (13) is less than ψ multiplied by the part of (11) inside the parenthesis, which I will call y_{int}. Thus our goal is to find a T such that for all later times the first term is less than ψy_{int}. I am not trying to prove convergence here. I am trying to find a loose bound for how long it takes the GDP to converge to the BGP. First I make my life easier by creating a looser bound which is easier to work with:

$$\begin{aligned}
\int_0^\infty x^{\frac{\eta-1}{\eta}} C e^{\alpha(1-\frac{\gamma}{\theta})t} \frac{1}{\theta} x^{\frac{1-\theta}{\theta}} e^{-\left(Ce^{\alpha(1-\frac{\gamma}{\theta})t} x^{\frac{1}{\theta}} + Dx^{\frac{1}{\gamma}}\right)} dx &\leq \int_0^\infty x^{\frac{\eta-1}{\eta}} C e^{\alpha(1-\frac{\gamma}{\theta})t} \frac{1}{\theta} x^{\frac{1-\theta}{\theta}} e^{-Dx^{\frac{1}{\gamma}}} dx \\
&= E e^{\alpha(1-\frac{\gamma}{\theta})t} \int_0^\infty v^{\frac{\eta+\eta\theta-\theta}{\eta\theta}\gamma - 1} e^{-v} dv \\
&= E e^{\alpha(1-\frac{\gamma}{\theta})t} \Gamma\left(\frac{\eta+\eta\theta-\theta}{\eta\theta}\gamma\right)
\end{aligned} \tag{15}$$

Where $E = \frac{1}{\theta} C D^{\frac{\theta-\eta\theta-\eta}{\eta\theta}}$. Now we find the right T:

$$\begin{aligned}
E e^{\alpha(1-\frac{\gamma}{\theta})t} \Gamma\left(\frac{\eta+\eta\theta-\theta}{\eta\theta}\gamma\right) &< \psi y_{int} \\
e^{\alpha(1-\frac{\gamma}{\theta})t} &< \frac{\psi y_{int}}{E\ \Gamma\left(\frac{\eta+\eta\theta-\theta}{\eta\theta}\gamma\right)} \\
t &> \frac{\ln \psi y_{int} - \ln E\ \Gamma\left(\frac{\eta+\eta\theta-\theta}{\eta\theta}\gamma\right)}{\alpha(1-\frac{\gamma}{\theta})} \equiv T
\end{aligned} \tag{16}$$

My notation has wandered far away from the fundamentals of the model. Throughout I assume that $\eta > \gamma$, which will be true for any reasonable calibration. E is basically the inverse of $\beta\mu\left(e^{\alpha\Delta} - 1\right)$, so that the smaller the intervention, the larger T must be. From the denominator, we see that the more similar γ and θ are, the longer we must wait. Likewise, it takes longer to converge if α, the number of meetings per period, is small.

Short-run Growth effects

Sensitivity to exposure to foreign tails might be significant in the long-run, but not produce much change in the short-run–insert Keynes quote. In this section, I analyze a simple example to show that limited exposure to foreign ideas can have effects in the relatively short term as well. Let there be two countries, one with a fat tail, and the other with a thin tail. The countries are parameterized as follows:

[4] I didn't formally prove this, but since the growth rate is monotonically increasing it is intuitively clear.

Parameter	Fat	Thin
α	0.05	0.05
θ	4	1
λ	100	1

I start the countries off with their long-run autarchic steady state distributions. With these parameters, the growth rates of Thin and Fat are respectively 5% and 20% annually. The experiment is to have Thin take its cost draws from fat for two years, and then isolate it again and examine its growth rate.

Years Post-Intervention	Thin Growth Rate
1	5.89%
2	5.94%
3	5.99%
4	6.04%
5	6.09%
6	6.15%
7	6.21%
8	6.27%
9	6.33%
10	6.40%

Within a few years, the effect of a short exposure to foreign technology has increased the Thin annual annual rate by 1.5%. This example is quite special, however, and generically it can take a long time for the effects of the tail to show up.

Conclusion

In analyzing the class of models like the one I consider in this note, researchers have been interested in the properties of balanced growth paths. In this note, these models have a tail-stealing feature. Even extremely limited access to outside technology is enough for each country to get the tail from the best country. This tail completely determines the long run growth properties of countries, so there is a strong convergence feature built into these models.

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